On the unsmoothing of functions on the real line

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Communicated by Prof. A.C. Zaanen at the meeting of November 25, 1985

SUMMARY

The smoothing $T_a f$ of a locally-integrable function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$(T_a f)(x)=\frac{1}{2a}\int_{-a}^{a} f(x+y)dy,$$

where $a>0$. We show that T_a maps the space of locally-integrable functions on $\mathbb{R}$, $L_{\mathrm{loc}}(\mathbb{R})$, onto the space $AC_{\mathrm{loc}}(\mathbb{R})$ of locally-absolutely continuous functions on $\mathbb{R}$. We construct a map $R_a: AC_{\mathrm{loc}}(\mathbb{R})\to L_{\mathrm{loc}}(\mathbb{R})$ such that $T_a\circ R_a = Id$ and such that R_a preserves the differentiability properties optimally. We also analyse T_a on some subspaces of $L_{\mathrm{loc}}(\mathbb{R})$ and study the continuity properties of T_a.

1. INTRODUCTION

Let L_{loc} denote the space of locally-integrable functions on the real line. For $f\in L_{\mathrm{loc}}$ and $a>0$, let

$$(T_a f)(x)=\frac{1}{2a}\int_{-a}^{a} f(x+y)dy, \quad x\in\mathbb{R}.$$

The function $T_a f$ is called the smoothing of f. The main aim of this paper is to analyse the following:

i) Characterize the kernel and the range of the map $T_a: L_{\mathrm{loc}}\to L_{\mathrm{loc}}$.

* Contribution to the research project RECONSTRUCTION, Ko 506/8-1, of the German Research Council (DFG), directed by Professor D. Kölzow, Erlangen.

ii) Find a right inverse R_a of T_a such that for a smooth g, $R_a(g)$ is optimally smooth.
iii) Analyse T_a on various subspaces of L_{loc}, such as L_1, L_{com}-functions of compact support, and W_1^k-Sobolev spaces.

In practical situations, T_af represents the smoothing (moving average or sliding mean) of the raw data f. The problem of reconstructing a function f from the smoothing T_af is called the unsmoothing problem. In the case when f is an integrable function with compact support, reconstruction formulas using two-sided Laplace-transform were obtained by Van der Pol in [2] and [3].

2. GENERAL PROPERTIES OF T_a

In the rest of the paper we shall not distinguish functions which agree a.e. on $\mathbb{R}$.

DEFINITION 2.1

A function $f:\mathbb{R}\to\mathbb{R}$ is called locally-absolutely continuous, if f is differentiable a.e., $f'\in L_{\text{loc}}$, and $f(x)=f(0)+\int_0^x f'(y)dy$ for all x.

We shall denote by AC_{loc} the space of all locally-absolutely continuous functions on $\mathbb{R}$.

The following lemma is obvious:

LEMMA 2.2

Let $f,g\in L_{\text{loc}}$ and $a>0$. Then $T_af=g$ iff the following holds:
i) $g\in AC_{\text{loc}}$
ii) $2a\, g'(x)=f(x+a)-f(x-a)$ *a.e.*
iii) $g(0)=\dfrac{1}{2a}\int_{-a}^{a} f(y)dy.$

PROPOSITION 2.3

Let $f\in L_{\text{loc}}$ and $a>0$. Then $T_af=0$ iff
i) $f(x)=f(x+2a)$ *a.e.*
ii) $T_af(0)=0.$

PROOF. Follows from lemma 2.2. Compare John [1], chapter VI.

COROLLARY 2.4

T_a is injective on a subspace X of L_{loc} if whenever f satisfies i) *and* ii) *of* 2.3, *then $f=0$. In particular T_a is injective on the following subspaces of L_{loc}:*
i) $X=L_p(\mathbb{R})$ *with* $1\le p<\infty$
ii) $X=\{f\in L_{\text{loc}}|f$ *vanishes at* $\infty\}$
iii) $X=\{f\in L_{\text{loc}}|f(x)=0$ *for* $|x|\le b\}$, $b\ge a$

REMARK 2.5

T_a is not injective on $X=\{f\in L_{\text{loc}}|f(x)=0$ for $|x|\le b\}$ if $b<a$. For this choose a $g\neq 0$ on $[-a,a]$ with $g(x)=0$ for $x\in[-b,b]$ and $\int_{-a}^{a} g(y)dy=0$.

Now define f to be the periodic extension of g over $\mathbb{R}$ with period 2a. Then $T_a f=0$ but $0\neq f\in X$.

PROPOSITION 2.6

i) *Let* $g\in AC_{\text{loc}}$ *be such that* $g(x)=0$ *for all* $x\leq c$, *where* $c\in\mathbb{R}$. *Then there exists a unique* $f\in L_{\text{loc}}$ *such that* $T_a f=g$ *and* $f(x)=0$ *for all* $x\leq c+a$. *Further*

$$(*)\qquad f(x)=\frac{d}{dx}\sum_{n=0}^{\infty} 2a\, g(x-2na-a) \text{ a.e.}$$

ii) *Let* $g\in AC_{\text{loc}}$ *be such that* $g(x)=0$ *for all* $x\geq c$, *where* $c\in\mathbb{R}$. *Then there exists a unique* $f\in L_{\text{loc}}$ *such that* $T_a f=g$ *and* $f(x)=0$ *for all* $x\geq c-a$. *Further*

$$f(x)=\frac{d}{dx}\sum_{n=0}^{\infty} -2a\, g(x+2na+a) \text{ a.e.}$$

PROOF. We shall prove i). Let $g\in AC_{\text{loc}}$ with $g(x)=0$ for every $x\leq c$. We define f by (*). Then f is well defined and $f(x)=0$ for $x\leq c+a$.

Further

$$f(x+a)-f(x-a)=\frac{d}{dx}\left[\sum_{n=0}^{\infty} 2a\, g(x-2na)-\sum_{n=0}^{\infty} 2a\, g(x-2na-2a)\right]=$$

$$=2a\, g'(x) \text{ (a.e.).}$$

Also clearly $\int_{-a}^{a} f(y)dy=2a\, g(0)$.

Thus by lemma 2.2, $T_a f=g$. The uniqueness of f follows from proposition 2.3.

PROPOSITION 2.7

$$T_a(L_{\text{loc}})=AC_{\text{loc}}.$$

PROOF. Clearly $T_a(L_{\text{loc}})\subseteq AC_{\text{loc}}$. To prove the equality, let $g\in AC_{\text{loc}}$. Define $h_a:\mathbb{R}\to\mathbb{R}$ by

$$h_a(x)=\begin{cases} 0 & \text{if } x\leq -a \\ 1 & \text{if } x\geq a \\ \dfrac{1}{2a}(a+x) & \text{if } -a<x<a, \end{cases}$$

and put $g_1=h_a g$, $g_2=(1-h_a)g$. Then g_1 and g_2 satisfy the conditions of proposition 2.6.

Thus there exist functions $f_1, f_2\in L_{\text{loc}}$ such that $T_a(f_i)=g_i$, $i=1,2$. Hence $f=f_1+f_2\in L_{\text{loc}}$ and $T_a f=g$.

COROLLARY 2.8

Let $g \in AC_{\text{loc}}$ and let $h \in L_{\text{loc}}$. There exists a (unique) $f \in L_{\text{loc}}$ such that $f(x) = h(x)$ for $x \in [-a, a]$ while $T_a f = g$ iff $T_a h(0) = g(0)$.

PROOF. Obviously if there exists an $f \in L_{\text{loc}}$ such that $T_a f = g$ and $f(x) = h(x)$ for $x \in [-a, a]$, then $T_a h(0) = g(0)$. Conversely, given $g \in AC_{\text{loc}}$ and $h \in L_{\text{loc}}$ such that $T_a h(0) = g(0)$, define $f_h(x) = h(x) - (R_a g)(x)$ for $x \in [-a, a]$ with some right inverse $R_a g$ of g and extend f_h periodically with period $2a$ over $\mathbb{R}$. Put $f = R_a g + f_h$. Then $T_a f = g$ and $f(x) = h(x)$ for $x \in [-a, a]$. The uniqueness of f follows from corollary 2.4 (iii).

REMARK 2.9

In view of propositions 2.6 and 2.7, given $g \in AC_{\text{loc}}$, one can construct an $f \in L_{\text{loc}}$ such that $T_a f = g$. Also, by proposition 2.3, any two such f's will differ only by a function ϕ which is periodic with period $2a$ and satisfies $\int_{-a}^{a} \phi(y)dy = 0$. If one replaces h_a in 2.7 by a C^∞-function $h_a^*: \mathbb{R} \to \mathbb{R}$ such that $h_a^*(x) = 1$ for $x \geqq a$ and $h_a^*(x) = 0$ for $x \leqq -a$, one gets a right inverse R_a^* of T_a such that

i) $R_a^*(C^{n+1}(\mathbb{R})) \subseteq C^n(\mathbb{R})$ for all $n \in \mathbb{N}$,

ii) $R_a^*(C^\infty(\mathbb{R})) \quad \subseteq C^\infty(\mathbb{R})$.

From this and lemma 2.2 it follows that $T_a(C^n(\mathbb{R})) = C^{n+1}(\mathbb{R})$ for all $n \in \mathbb{N}$, and $T_a(C^\infty(\mathbb{R})) = C^\infty(\mathbb{R})$.

Thus R_a^* preserves differentiability properties optimally.

Next we describe the relation between the supports of $T_a f$ and f in the case $f \in L_p$ $(1 \leq p < \infty)$. Of course, the same is true for other subspaces of L_{loc}.

PROPOSITION 2.10

For $f \in L_p$, $1 \leq p < \infty$ and $R \geq 0$ the following statements are equivalent:

i) $(T_a f)(x) = 0$ *for* $|x| \geq R + a$

ii) $f(x) = 0 \qquad$ *for* $|x| \geq R$.

PROOF. Obviously ii) implies i). So suppose $f \in L_p$, $R \geq 0$ and $(T_a f)(x) = 0$ for $|x| \geq R + a$.

Put $f_1(x) = f(x)$ for $x \in [-R, R]$ and $f_1(x) = 0$ otherwise. Put also $f_2 = f - f_1$. Then $f_2(x) = f_2(x - 2a)$ for $x < -R$ and $f_2(x) = f_2(x + 2a)$ for $x > R$. But $f_2 \in L_p$ and hence $f_2 = 0$.

3. SMOOTHING FOR L_1-FUNCTIONS

We showed in corollary 2.4 that T_a is injective on L_p for all $1 \leq p < \infty$. In fact T_a is also injective on the following spaces:

$$L_p^l = \{f \in L_{\text{loc}} | \chi_{(-\infty, c]} f \in L_p \text{ for all } c \in \mathbb{R}\},$$

$$L_p^r = \{f \in L_{\text{loc}} | \chi_{[c, \infty)} f \in L_p \text{ for all } c \in \mathbb{R}\}.$$

In the following we treat only the case $p=1$. We begin with the class $L_{\text{com}} = \{f \in L_{\text{loc}} | f$ has compact support$\}$.

PROPOSITION 3.1

For $g \in AC_{\text{loc}}$, the following statements are equivalent:

i) *$T_a f = g$ for some $f \in L_{\text{com}}$*

ii) *g has compact support and $\sum_{n \in \mathbb{Z}} g(x+2na)$ is a constant independent of x.*

PROOF. Let $f \in L_{\text{com}}$ and $T_a f = g$. Then g has compact support and

$$\sum_{n \in \mathbb{Z}} g(x+2na) = \frac{1}{2a} \int f(y) dy$$

is independent of x. This proves i) implies ii). Now suppose $g \in AC_{\text{loc}}$ has compact support and $\sum_{n \in \mathbb{Z}} g(x+2na)$ is a constant. We define

$$f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} 2ag(x-2na-a) = \frac{d}{dx} \sum_{n=0}^{\infty} -2ag(x+2na+a).$$

Because of our conditions on g, f is well defined and $f \in L_{\text{com}}$. It follows from proposition 2.6 that $T_a f = g$. Hence ii) implies i).

PROPOSITION 3.2

i) *Let*

$$A_r = \{g \in AC_{\text{loc}} | g \in L_1^r, g' \in L_1^r, \text{ and } \sum_{n=0}^{\infty} g'(x+2na) \text{ converges in } L_1^r\}.$$

Then T_a maps L_1^r injectively onto A_r and $T_a^{-1}: A_r \to L_1^r$ is given by

$$(T_a^{-1} g)(x) = \frac{d}{dx} \sum_{n=0}^{\infty} -2ag(x+2na+a).$$

ii) *Let*

$$A_l = \{g \in AC_{\text{loc}} | g \in L_1^l, g' \in L_1^l, \text{ and } \sum_{n=0}^{\infty} g'(x-2na) \text{ converges in } L_1^l\}.$$

Then T_a maps L_1^l injectively onto A_l and $T_a^{-1}: A_l \to L_1^l$ is given by

$$(T_a^{-1} g)(x) = \frac{d}{dx} \sum_{n=0}^{\infty} 2ag(x-2na-a).$$

PROOF. We prove i). The proof of ii) can be given similarly. Let $f \in L_1^r$ and $g = T_a f$. Then $g \in AC_{\text{loc}}$ and for any $c \in \mathbb{R}$ we have

$$\int_c^{\infty} |g(x)| dx = \int_c^{\infty} \left| \frac{1}{2a} \int_{x-a}^{x+a} f(y) dy \right| dx \leq$$

$$\leq \frac{1}{2a} \int_c^{c+2a} \int_{x-a}^{c+a} |f(y)| dy dx + \int_{c+a}^{\infty} |f(y)| dy < \infty.$$

Hence $g \in L_1^r$. Since $f \in L_1^r$ and $2ag'(x) = f(x+a) - f(x-a)$ we have $g' \in L_1^r$. Further we have

$$f(x) = f(x+2sa+a) + \sum_{n=0}^{s} -2ag'(x+2na+a)$$

with

$$\int_c^\infty |f(x+2sa+a)|dx \to 0 \text{ as } s \to \infty$$

and therefore $\sum_{n=0}^\infty g'(x+2na+a)$ converges in L_1^r. Now let $g \in A_r$ and $F(x) = \sum_{n=0}^\infty -2ag(x+2na+a)$. Then, by Lebesgue's theorem, $F \in AC_{\text{loc}}$ and $F'(x) = \sum_{n=0}^\infty -2ag'(x+2na+a)$ (a.e.). We have $F' \in L_1^r$ and

$$(T_a F')(x) = \frac{1}{2a} \int_{x-a}^{x+a} F'(y)dy = \frac{1}{2a}(F(x+a) - F(x-a)) = g(x).$$

This proves the proposition.

PROPOSITION 3.3

$T_a(L_1) = \{g \in AC_{\text{loc}} | g \in L_1, g' \in L_1$, and $\sum_{n=0}^\infty g'(x+2na)$ converges in L_1^r and the sum is in $L_1\}$, and the inverse is given by

$$(T_a^{-1}g)(x) = \frac{d}{dx} \sum_{n=0}^\infty -2ag(x+2na+a) = \frac{d}{dx} \sum_{n=0}^\infty 2ag(x-2na-a) \text{ a.e.}$$

(where $g \in T_a(L_1)$).

PROOF. Follows from proposition 3.2.

REMARK 3.4

It is easy to see that $T_a^{-1}: T_a(L_1) \to L_1$ is not continuous when $T_a(L_1)$ and L_1 have L_1-topology or pointwise convergence topology (a.e.). In other words, the unsmoothing problem is illposed. For the numerical treatment of such problems one has to apply regularization techniques. We shall show (corollary 3.6) that the unsmoothing problem is well-posed on certain Banach spaces. It is not difficult to carry over proposition 3.3 to the spaces $W_1^k(\mathbb{R})$; the result is the following:

PROPOSITION 3.5

$T_a(W_1^k(\mathbb{R})) = \{g \in W_1^{k+1}(\mathbb{R}) | \sum_{n=0}^\infty g'(x+2na)$ converges in L_1^r and the sum is in $W_1^k(\mathbb{R})\}$.

COROLLARY 3.6

The map T_a is a Banach space isomorphism between $W_1^k(\mathbb{R})$ $(k \in \mathbb{N}_0)$ and $T_a(W_1^k(\mathbb{R}))$ if one considers the norm

$$\|g\| := \|g\|_{k,1} + \| \sum_{n=0}^\infty g'(x+2na)\|_{k,1}$$

on $T_a(W_1^k(\mathbb{R}))$. Here, $\|g\|_{k,1}=\sum_{i=0}^{k}\|g^{(i)}\|_1$ denotes the usual norm on $W_1^k(\mathbb{R})$. Thus, the unsmoothing map

$$(T_a^{-1}g)(x)=\frac{d}{dx}\sum_{n=0}^{\infty}-2ag(x+2na+a)$$

is continuous for those topologies.

REMARK 3.7

i) In [1], F. John relates the map T_a to the solution $u(x,t)$ of the wave equation $u_{xx}=u_{tt}$, under the initial value conditions $u(x,0)=0$, $u_t(x,0)=f(x)$, by showing $u(x,t)=t\cdot T_t f(x)$ for all $t>0$. From this, F. John derives the following reconstruction formula for T_1 (see John [1], (6.35) for the three-dimensional analog):

$$f(x)=-2\sum_{n=0}^{\infty}\frac{d^2}{dx^2}((2n+1)(T_{2n+1}g)(x))$$

if $T_1 f=g$, and g fullfills certain regularity conditions.

ii) Concerning uniqueness, John [1] proved the following result: Let

$$\tilde{f}\in C(\mathbb{R}^3) \text{ and } \tilde{g}(x)=\frac{1}{4\pi}\int_{|y|=1}\tilde{f}(x+y)d\omega(y).$$

Then, $\tilde{f}$ is uniquely determined by $\tilde{g}$ and by the values of $\tilde{f}$ in the sphere $|x|<1+\varepsilon$, for every $\varepsilon>0$ (but not for $\varepsilon=0$).

For $f\in C(\mathbb{R})$, define $\tilde{f}\in C(\mathbb{R}^3)$ by $\tilde{f}(x_1,x_2,x_3)=f(x_1)$. Then, the formula $(T_1 f)(x)=\tilde{g}((x,0,0))$ holds, where $\tilde{g}$ is defined as above (see John [1], (1.1)). By this, it follows that $f\in C(\mathbb{R})$ is determined by $T_1 f$ and the values of f in some interval of length $2+\varepsilon$. As it is shown in corollary 2.4, this result even holds for $\varepsilon=0$.

iii) The first author carried over some of the results obtained above to higher dimensions: unsmoothing a function when all integrals $\int_{A+x}f(y)d\lambda^n$, $x\in\mathbb{R}^n$, A being a cube are given. This can be done by a repeated application of one-dimensional unsmoothing.

LITERATURE

1. John, F. – Plane waves and spherical means applied to partial differential equations. New York: Interscience, 1955.
2. Pol, B. van der – Smoothing and "Unsmoothing". In: M. Kac, Probability and related topics in physical sciences. New York: Interscience, 1959.
3. Pol, B. van der and H. Bremmer – Operational calculus. Cambridge: Cambridge University Press, 1964.